$\hat{\mathbb{Z}}^{\wedge} = \hat{\mathbb{Z}}$

$\Delta_X \twoheadrightarrow \hat{\mathbb{Z}}$

$\Delta_X^{\Theta} := \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]]$

ϑ $\quad [\,,\,]$: comm. rel

§7. Étale Theta Functions — Three Rigidities

§7.1 Theta-Related Varieties

$K/\mathbb{Q}_p$ fin. $\subset \overline{K}$, $G_K := \mathrm{Gal}(\overline{K}/K)$

$\mathfrak{X} \to \mathrm{Spf}\, \mathcal{O}_K$ stable curve of type $(1,1)$ s.t.
special fiber is singular, geom. irred.
the node is rational.
Raynaud gen. fiber is smooth.
X

marked pt, special fiber

$\uparrow$

log str.

Π $\quad \wr \quad$ $\log\text{-}\Pi$

$$\Pi_X^{tmp} \supset \Delta_X^{tmp}$$

$$\Pi_X := (\Pi_X^{tmp})^\wedge, \quad \Delta_X := (\Delta_X^{tmp})^\wedge$$

$$\Delta_X^{tmp} \twoheadrightarrow \underline{\mathbb{Z}}$$

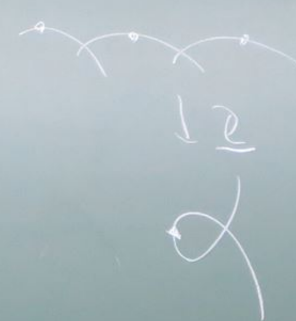
comm. univ. cov.

$$\underline{\mathbb{Z}}^\wedge =: \underline{\hat{\mathbb{Z}}}$$

$$\Delta_X \twoheadrightarrow \underline{\hat{\mathbb{Z}}}$$

$$\Delta_X^{\Theta} := \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]]$$

ϑ $\qquad [\ ,\ ]$: comm. rel



§7. Étale Theta Functions — Three Rigidities

$\mathbb{Q}$

ϑ

θ

$\ominus$ Θ

O

$$\Delta_\Theta := \Lambda^2 \Delta_X^{ab} \quad (\cong \hat{\mathbb{Z}}(1))$$

$$\Delta_X^{ell} := \Delta_X^{ab}$$

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{ell} \to 1$$

$$1 \to \hat{\mathbb{Z}}(1) \to \Delta_X^{ell} \to \underline{\hat{\mathbb{Z}}} \to 1$$

$$\begin{tikzcd}
\Delta_X \arrow[r, two heads] & \Delta_X^\Theta \arrow[r, two heads] & \Delta_X^{ell} \\
\Delta_X^{top} \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{top})^\Theta \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{top})^{ell} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Δ_X^Θ

theta lies in

anabelian étale

properties of theta

are deduced

from Π_X

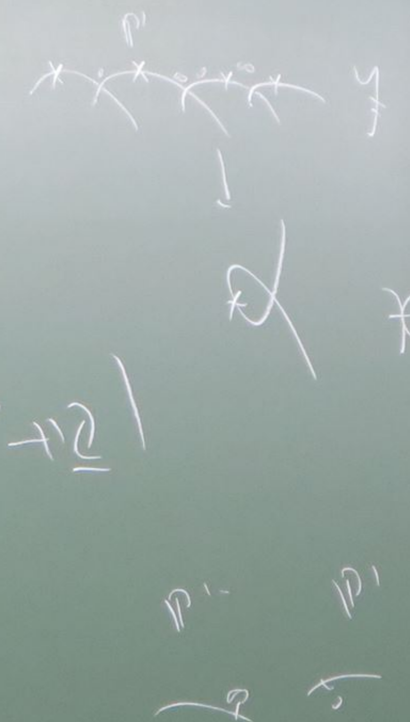
$$
\begin{array}{ccccc}
\Pi_X^{tp} & \twoheadrightarrow & (\Pi_X^{tp})^{\Theta} & \twoheadrightarrow & (\Pi_X^{tp})^{ell} \\
\cup & & \cup & & \cup \\
\Delta_X^{tp} & \twoheadrightarrow & (\Delta_X^{tp})^{\Theta} & \twoheadrightarrow & (\Delta_X^{tp})^{ell}
\end{array}
$$

$$1 \to \Delta_{\Theta} \to (\Delta_X^{tp})^{\Theta} \to (\Delta_X^{tp})^{ell} \to 1$$

$$1 \to \hat{\mathbb{Z}}(1) \to (\Delta_X^{tp})^{ell} \to \underline{\ell} \to 1$$

$$Y \twoheadrightarrow X \quad (\text{resp. } \underline{Y} \to \underline{X}) \leadsto \Pi_Y^{tp} := \mathrm{Ker}(\Pi_X^{tp} \twoheadrightarrow \underline{\ell})$$

$$\mathrm{Gal}(Y/X) = \underline{\ell}$$



$$
\begin{array}{ccccc}
\Delta^{tp}_X & \twoheadrightarrow & (\Delta^{tp}_X)^{\Theta} & \twoheadrightarrow & (\Delta^{tp}_X)^{ab} \\
\cup & & \cup & & \cup \\
\Delta^{tp}_Y & \twoheadrightarrow & (\Delta^{tp}_Y)^{\Theta} & \twoheadrightarrow & (\Delta^{tp}_Y)^{ab}
\end{array}
\qquad
\begin{array}{ccccc}
\Pi^{tp}_X & \twoheadrightarrow & (\Pi^{tp}_X)^{\Theta} & \twoheadrightarrow & (\Pi^{tp}_X)^{ab} \\
\cup & & \cup & & \cup \\
\Pi^{tp}_Y & \twoheadrightarrow & \underline{(\Pi^{tp}_Y)^{\Theta}} & \twoheadrightarrow & \underline{(\Pi^{tp}_Y)^{ab}}
\end{array}
$$

$$1 \to \underset{\substack{\text{abelian} \\ (\cong \hat{\mathbb{Z}}(1)^{\oplus 2})}}{\Delta_{\Theta}} \to (\Delta^{tp}_Y)^{\Theta} \to \underset{(\cong \hat{\mathbb{Z}}(1))}{(\Delta^{tp}_Y)^{ab}} \to 1$$

$q_X \in \mathcal{O}_K$: q-parameter of X

$N \geq 1$, $K_N := K(\mu_N, q_X^{1/N}) \subset \bar{K}$

any decomp. gp of a cusp of Y $\rightsquigarrow G_K \to (\Pi_Y^{tp})^{ell}$

restriction

well-def up to $(\Delta_Y^{tp})^{ell}$-conj.

$$G_{K_N} \hookrightarrow \underbrace{G_K \to (\Pi_Y^{tp})^{ell} \twoheadrightarrow}_{\text{inj}} (\Pi_Y^{tp})^{ell} / N (\Delta_Y^{tp})^{ell}$$

def K_N

image is stable under Π_X^{tp}-conj.

$$1 \to \mathbb{Z}/N\mathbb{Z}(1) \to (\Delta_X^{tp})^{ell} / N (\Delta_Y^{tp})^{ell} \to \underline{\ell} \to 1$$

triv

G_{K_N}

Δ_X^{Θ}

$\Delta^{tp} \to (\Delta^{tp})^{\Theta} \to (\Delta^{tp})^{ell}$ $\quad \Pi_X^{tp} \to (\Pi_X^{tp})^{\Theta} \to (\Pi_X^{tp})^{ell}$

$\mathcal{Y}_N \to \mathcal{Y}$ normalisation of $\mathcal{Y}$ in Y_N

$\rightsquigarrow G_{K_N} \hookrightarrow (\Pi_Y^{tmp})^{ell} / N(\Delta_Y^{tmp})^{ell}$

ring $\rightsquigarrow Y_N \to Y$

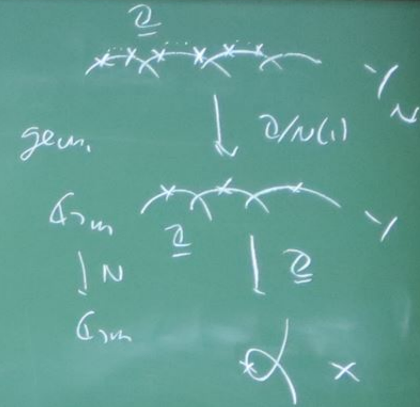
Galois cov.

$1 \to \Pi_{Y_N}^{tmp} \to \Pi_Y^{tmp} \to Gal(Y_N/Y) \to 1$

$1 \to (\Delta_Y^{tmp})^{ell} \otimes \mathbb{Z}/N \to Gal(Y_N/Y) \to Gal(K_N/K) \to 1$

$(\cong \mathbb{Z}/N\mathbb{Z}(1))$

$\left(\begin{array}{l} \quad \mathbb{Z}/N(1) \\ \quad\ \cup \\ 1 \to \ker \to Gal(K_N/K) \to (\mathbb{Z}/N)^\times \end{array} \right)$



$$\begin{array}{ccccc}
\Delta_Y^{tp} & \twoheadrightarrow & (\Delta_Y^{tp})^{\Theta} & \twoheadrightarrow & (\Delta_Y^{tp})^{ell} \\
\cup & & \cup & & \cup \\
\Delta_{Y_N}^{tp} & \twoheadrightarrow & (\Delta_{Y_N}^{tp})^{\Theta} & \twoheadrightarrow & (\Delta_{Y_N}^{tp})^{ell}
\end{array}$$

$$\begin{array}{ccccc}
\Pi_Y^{tp} & \twoheadrightarrow & (\Pi_Y^{tp})^{\Theta} & \twoheadrightarrow & (\Pi_Y^{tp})^{ell} \\
\cup & & \cup & & \cup \\
\Pi_{Y_N}^{tp} & \twoheadrightarrow & (\Pi_{Y_N}^{tp})^{\Theta} & \twoheadrightarrow & (\Pi_{Y_N}^{tp})^{ell}
\end{array}$$

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \to (\Pi_{Y_N}^{tp})^{\Theta} / N (\Delta_Y^{tp})^{\Theta} \to G_{K_N} \to 1$$

$$(\cong \mathbb{Z}/N\mathbb{Z}(1))$$

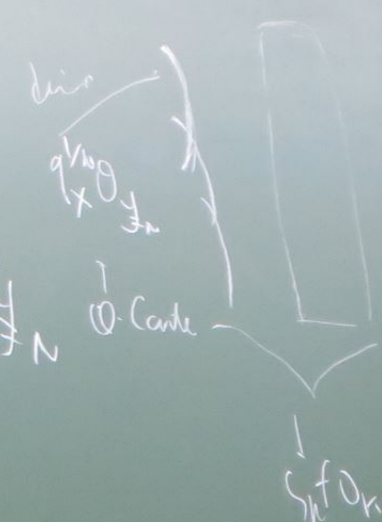
$\mathcal{Y}_N \twoheadrightarrow \mathcal{Y}$ normalisation of $\mathcal{Y}$ in Y_N

Choose an irred. comp. of $\mathcal{Y}$ as a "basepoint"

$\rightsquigarrow$ irred. comps. labelled by $\mathbb{Z}$

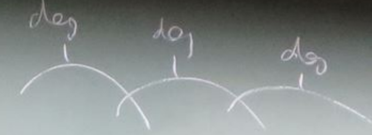
$$\overset{\deg}{\rightsquigarrow} \mathrm{Pic}(\mathcal{Y}_N) \xrightarrow{\sim} \mathbb{Z}^{\mathbb{Z}}$$

Cartier divisors on $\mathcal{Y}_N$



$\mathcal{L}_N ; \sim \mathbb{Q} \to \mathbb{Q}$

$\quad\quad\quad\; a \mapsto 1$



$\Gamma(Y_N, \mathcal{O}_{Y_N}) = \mathcal{O}_{K_N}$

$J_N := K_N(a^{1/N} \mid a \in K_N) \subset \overline{K}$

$\cup$ fin. Gal.

K_N

$1 \to \Delta_\Theta \otimes \mathbb{Z}/N \to (\Pi^{tp}_{Y_N})^{\Theta} / N(\Delta^{tp}_{Y})^{\Theta} \to G_{K_N} \to 1$

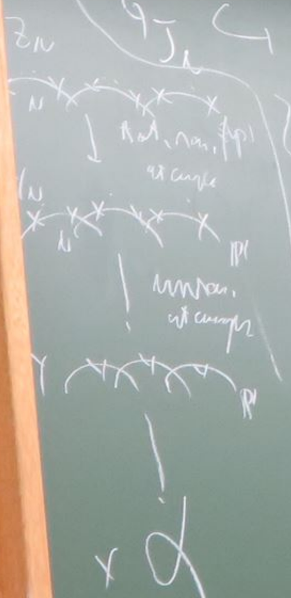
two splittings $\rightsquigarrow H^1(G_{K_N}, \Delta_\Theta \otimes \mathbb{Z}/N)$

$\downarrow$

def of J_N : $\quad H^1(G_{J_N}, \quad " \quad)$

$\quad\quad\quad\quad\quad\quad 0$

splittings coincide on G_{J_N}



$$G_{J_N} \hookrightarrow (\Pi^{tp}_{Y_N})^{\Theta} / N(\Delta^{tp}_{Y})^{\Theta}$$

stable, under conj. by Π^{tp}_X

$Z_N \to Y_N$ fin. Gal.

$$1 \to \Pi^{tp}_{Z_N} \to \Pi^{tp}_{Y_N} \to Gal(Z_N/Y_N) \to 1$$

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N \to Gal(Z_N/Y_N) \to Gal(J_N/K_N) \to 1$$

$$\begin{array}{ccccc} \Delta^{tp}_{Y_N} & \to & (\Delta^{tp}_{Y_N})^{\Theta} & \to & (\Delta^{tp}_{Y_N})^{\mathrm{ell}} \\ \cup & & \cup & & \cup \\ \Delta^{tp}_{Z_N} & \to & (\Delta^{tp}_{Z_N})^{\Theta} & \to & (\Delta^{tp}_{Z_N})^{\mathrm{ell}} \end{array}$$

$$\begin{array}{ccccc} \Pi^{tp}_{Y_N} & \to & (\Pi^{tp}_{Y_N})^{\Theta} & \to & (\Pi^{tp}_{Y_N})^{\mathrm{ell}} \\ \cup & & \cup & & \cup \\ \Pi^{tp}_{Z_N} & \to & (\Pi^{tp}_{Z_N})^{\Theta} & \to & (\Pi^{tp}_{Z_N})^{\mathrm{ell}} \end{array}$$

$\mathfrak{Z}_N \to \mathfrak{Y}_N$ normalization of $\mathfrak{Y}$ in Z_N

$s_1 \in \Gamma(\mathfrak{Y}, \mathcal{L}_1)$ section zero locus = cusps $(\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = \mathcal{O}_K)$

well-def'd up to $\mathcal{O}_K^\times$-multiple

$\mathcal{L}_N^{\otimes N} \cong \mathcal{L}_1|_{\mathfrak{Y}_N}$ $\quad$ $\mathrm{Gal}(Y/X) \curvearrowright \mathcal{L}_1 \rightsquigarrow \mathrm{Gal}(Y_N/X) \curvearrowright \mathcal{L}_1|_{\mathfrak{Y}_N}$

Fix $\sim$ identify $\quad$ $= ?$ $\quad$ unique preserving s_1

Lem 7.1 ([EtTh, Prop 1.1])

(1). $s_1|_{Y_N} \in \Gamma(Y_N, \mathcal{L}_1|_{Y_N}) = \Gamma(Y_N, \mathcal{L}_N^{\otimes N})$

has an N-th root $s_N \in \Gamma(\mathfrak{Z}_N, \mathcal{L}_N|_{\mathfrak{Z}_N})$

(2). $\exists$ unique action $\Pi_X^{tp} \curvearrowright \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$ over $Y_N \times_{O_{K_N}} O_{J_N}$

which is compat w/ section $s_N : \mathfrak{Z}_N \longrightarrow \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$

factors through $\Pi_X^{tp} \twoheadrightarrow \Pi_X^{tp} / \Pi_{Z_N}^{tp} = \mathrm{Gal}(Z_N/X)$

$\Delta_X^{tp} / \Delta_{Z_N}^{tp} \curvearrowright \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$ faithful

s^{gal}_{θ} s^{Θ}

mono s^{Θ}

$\ddot{K}_N := K_{2N}$, $\ddot{J}_N := \ddot{K}_N(a^{1/N} \mid a \in \ddot{K}_N) \subset \bar{K}$,

$\ddot{Y} := Y_2$

$$\ddot{\mathcal{Y}}_N := \mathcal{Y}_{2N}, \quad \ddot{Y}_N := Y_{2N} \times_{\ddot{K}_N} \ddot{J}_N,$$

$$\ddot{\mathcal{L}}_N := \mathcal{L}_N|_{\ddot{\mathcal{Y}}_N} = \mathcal{L}_{2N}^{\otimes 2} \otimes_{O_{\ddot{K}_N}} O_{\ddot{J}_N}$$

$\ddot{Z}_N$: the composite of

$$\begin{array}{ccc} \ddot{①} & \to & Z_N \\ \downarrow & & \downarrow \\ \ddot{Y}_N & \to & Y_N \end{array}$$

$\ddot{\mathfrak{Z}}_N$: normalization of $\mathfrak{Z}_N$ in $\ddot{Z}_N$

$$\ddot{Y} := \ddot{Y}_1 = Y_2, \quad \ddot{\mathcal{Y}} := \ddot{\mathcal{Y}}_1 = \mathcal{Y}_2$$

$$\ddot{K} := \ddot{K}_1 = \ddot{J}_1 = K_2$$

$\Pi^{top}_X \curvearrowright \overset{''}{\mathcal{Y}}_{\mathfrak{X}_N}, \mathcal{Y}_{\mathfrak{X}_N}, \mathcal{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{\mathfrak{X}_N}$ (compat.)

$\rightsquigarrow \Pi^{top}_X \curvearrowright \overset{''}{\mathcal{L}}_N \curvearrowleft \Pi^{top}_X / \Pi^{top}_{\mathfrak{X}_N}$

\mathcal $\mathcal{L}$ || \mathfrak $\mathfrak{L}$

Choose an orientation of the dual graph of $\mathcal{Y}_{\mathfrak{X}}$

$\rightsquigarrow \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$

$\mathcal{Y}_N$

$(\overset{''}{\mathcal{Y}}_N)_j$

node

compl. divisor j-th comp.

$\mathrm{Spf}\, \mathcal{O}_{\mathfrak{X}_N}[[u,v]]/(uv - q^{1/2N}_{X})$

"$q^{1/2N}_X = 0$"

Put $\mathfrak{D}_N := \sum_{j\in\mathbb{Z}} j^2 (\ddot{\mathcal{Y}}_N)_j$

j-th cp "$q_x^{j^2/2N} = 0$"

Claim $\mathcal{O}_{\ddot{\mathcal{Y}}_N}(\mathfrak{D}_N) \cong \ddot{\mathcal{L}}_N \quad \left(\cong \mathcal{L}_{2N}^{\otimes 2} \otimes_{\mathcal{O}_{\ddot{Y}_N}} \mathcal{O}_{\ddot{\mathcal{Y}}_N}\right)$

☺ $\mathrm{Pic}(\ddot{\mathcal{Y}}_N) \cong \mathbb{Z}^2$

$\rightsquigarrow$ it suffices to show $\mathfrak{D}_N \cdot (\ddot{\mathcal{Y}}_N)_i = 2 \quad \forall i \in \mathbb{Z}$

inter. prod.

$0 = \ddot{\mathcal{Y}}_N \cdot (\ddot{\mathcal{Y}}_N)_i = \sum_{j\in\mathbb{Z}} (\ddot{\mathcal{Y}}_N)_j \cdot (\ddot{\mathcal{Y}}_N)_i = 2 + \left((\ddot{\mathcal{Y}}_N)_i\right)^2$

$\Rightarrow \left((\ddot{\mathcal{Y}}_N)_i\right)^2 = -2$

$\Rightarrow (\ddot{\mathcal{Y}}_N)_i \cdot$

$\Rightarrow \mathfrak{D}_N \cdot (\ddot{\mathcal{Y}}_N)_i = \sum_{j\in\mathbb{Z}} j^2 (\ddot{\mathcal{Y}}_N)_j \cdot (\ddot{\mathcal{Y}}_N)_i = (i-1)^2 + (i+1)^2 - 2i^2 = 2$ ⌟

$\rightsquigarrow \tau_N : \ddot{\mathcal{Y}}_N \longrightarrow \ddot{\mathcal{L}}_N$

$\uparrow$ zero locus $= \mathfrak{D}_N$ well-def up to $\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_N}$-multiple

theta trivialization

$\Pi^{\text{tmp}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{Y}}_N, \ddot{\mathcal{L}}_N$

preserves τ_N up to $\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_N}$-multiple $\left(\begin{array}{l} \because \Pi^{\text{tmp}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{Y}}_N \\ \text{fixes } \mathfrak{D}_N \end{array} \right)$

$M \geq 1, \quad M \mid N$

$$\mathcal{Y}_N \to \mathcal{Y}_M \to \mathcal{Y}$$

$$\ddot{\mathcal{Y}}_N \to \ddot{\mathcal{Y}}_M \to \ddot{\mathcal{Y}} \qquad \mathcal{L}_M|_{\mathcal{Y}_N} \cong \mathcal{L}_N^{\otimes(N/M)}$$

$$\mathfrak{Z}_N \to \mathfrak{Z}_M \to \mathcal{Y} \qquad \ddot{\mathcal{L}}_M|_{\ddot{\mathcal{Y}}_N} \cong \ddot{\mathcal{L}}_N^{\otimes(N/M)}$$

$$\mathrm{Aut}\, \ddot{\mathcal{Y}}_N \curvearrowright \begin{array}{ccc} \ddot{\mathcal{L}}_N & \to & \ddot{\mathcal{L}}_M \\ \downarrow \tau_N & \curvearrowright & \downarrow \tau_M \\ \ddot{\mathcal{Y}}_N & \to & \ddot{\mathcal{Y}}_M \end{array}$$

up to $\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_N}$-mult at $\ddot{\mathcal{L}}_N$

$\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_M}$-mult at $\ddot{\mathcal{L}}_M$

---

th cup "$q_x^{j^2/2N} = 0$"

$$\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U}) \quad (\text{by } 7.4\,(2),(3))$$

def of $\ddot{\mathfrak{J}}_{N}$, $\rightsquigarrow$ $\ddot{\mathcal{L}}_N \to \ddot{\mathcal{L}}_M$ $\quad \mathcal{L}_M|_{\ddot{y}_N} = \mathcal{L}_N$

$\tau_N \quad \curvearrowright \tau_M$ up to $\mathcal{O}^\times_{\ddot{\mathfrak{J}}_N}$-mult at $\ddot{\mathcal{L}}_N$

$\ddot{y}_N \to \ddot{y}_M$ $\quad \mathcal{O}^\times_{\ddot{\mathfrak{J}}_M}$-mult at $\ddot{\mathcal{L}}_M$

$^2/2N = 0$"

$1^2 \quad 2^2$

$\boxed{\mathfrak{s}^{\mathrm{alg}}}$ $\boxed{\mathfrak{s}^{\Theta}}$

$(i-1)^2 + (i+1)^2$

$2i^2 = 2$

$\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$ (Lem 7.4 (2), (3))

$\ddot{Y}$

$\}$ can choose τ_1 so that

$\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{L}}_1$ preserves $\pm\tau_1$

In summary by def of $\ddot{\mathfrak{J}}_N$

- By modifying τ_N's by $\mathcal{O}^\times_{\ddot{\mathfrak{J}}_N}$-multiple $\rightsquigarrow$ may assume $\tau_N^{N/M} = \tau_M$ for $M|N$
- In particular, compat. sys. of actions of $\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \{\ddot{y}_N\}_{N\geq 1}, \{\ddot{\mathcal{L}}_N\}_{N\geq 1}$
- each action $\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \ddot{y}_N, \ddot{\mathcal{L}}_N$ which preserves $\{\tau_N\}_{N\geq 1}$ differs from the action det'd by $\Pi^{\mathrm{tp}}_{Y} \curvearrowright y_N, \mathcal{L}_N \otimes \mathcal{O}_{\ddot{\mathfrak{J}}_N}$ by an elt of $\mu_N(\ddot{\mathfrak{J}}_N)$ in Lem 7.1 (2)

<u>Def 7.2</u> Take ϵ_N's as above.

Taking the difference of the cusp. ext. sys. of the action of $\Pi^{tp}_{Y} \curvearrowright \{\ddot{Y}_N\}_{N\geq 1}$, $\{\ddot{\mathcal{Z}}_N\}_{N\geq 1}$

in Lem 7.1 det'd by $\{s_N\}_{N\geq 1}$

and the cusp. ext. sys. of the action $\Pi^{tp}_{Y} \curvearrowright \{\ddot{Y}_N\}_{N\geq 1}, \{\ddot{\mathcal{Z}}_N\}_{N\geq 1}$ det'd by $\{\epsilon_N\}_{N\geq 1}$

$\rightsquigarrow$ cohom. class $\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{Y}, \Delta_\Theta)$

via isom $\mu_N(\ddot{J}_N) \cong \Delta_\Theta \otimes \mathbb{Z}/N$

isom. scheme theory

Rem 7.21 (cf. [E+Th, Prop 1.3])

(1) η^{Θ} arises from a class in $\varprojlim_N H^1(\Pi^{tp}_{\ddot{Y}}/\Pi^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N)$

$$
\begin{array}{rl}
\eta^{\Theta} \in & \varprojlim_N H^1(\Pi^{tp}_{\ddot{Y}}/\Pi^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
 & \downarrow \\
 & \varprojlim_N H^1(\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
 & \cong \\
 & \varprojlim_N \mathrm{Hom}(\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
 & \in \\
 & \text{natural isom } \{\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N} \xrightarrow{\sim} \Delta_\Theta \otimes \mathbb{Z}/N\}_{N \geq 1}
\end{array}
$$

(2). $\varsigma_2 : \ddot{Y} \to \ddot{Z}$, well-def. up to $\mathcal{O}^\times_{\ddot{K}}$-multiple.

$\varsigma_{2N} : \ddot{Y}_N \to \ddot{Z}_N$: N-th root of ς_2,

$\tau_1 : \ddot{Y} \to \ddot{Z}$, well-def. up to $\mathcal{O}^\times_{\ddot{K}}$-multiple

$\tau_N : \ddot{Y}_N \to \ddot{Z}_N$: N-th root of τ_1

$\rightsquigarrow \ddot{\eta}^\Theta \in H^1(\Pi^{tmp}_{\ddot{Y}}, \Delta_\Theta)$ well-def. up to an $\mathcal{O}^\times_{\ddot{K}}$-multiple.

$\rightsquigarrow \mathcal{O}^\times_{\ddot{K}} \ddot{\eta}^\Theta \subset H^1(\Pi^{tmp}_{\ddot{Y}}, \Delta_\Theta)$

subset, indep. of the choices of ς_N's, τ_N's

We call any elt étale theta class

§7.2 Étale Theta Function

We recall the notions assoc. to a tangential base pt (cf. [AbsSect, Def 4.1(iii)] & before Def 4.1)

cusp $y \in \ddot{Y}(L)$ $L/\ddot{K}_i$ fin.

$D_y \subset \Pi_{\ddot{Y}}$ cusp. decomp. gp of y (well-def up to conj)

$$1 \to I_y \to D_y \to G_L \to 1$$

$(\cong \hat{\mathbb{Z}}(1))$ $\mathrm{Sect}(D_y \to G_L)$: the set of splittings of up to conj. by I_y

$\curvearrowleft$ torsor $H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$

(2) $S : \ddot{Y}, \ddot{\mathcal{Y}}$ well-def ...

ω_y : cotang. of $\ddot{Y}$ at y

$0 \neq \theta \in \omega_y$

take a sys. of N-th roots of any local coord. $t \in m_{\ddot{Y},y}$ w/ $dt|_y = \theta$

$\rightsquigarrow$ $\hat{\mathbb{Z}}(1)$ $(\cong I_y)$-torsor $\left(\ddot{Y}|^{\wedge}_y(t^{1/N})\right)_{N \geq 1} \to \ddot{Y}|^{\wedge}_y$ completion at y

$\rightsquigarrow$ $\mathrm{Sect}(D_y \to G_L)$ $\quad D_y \to I_y$ $\rightsquigarrow$ $\in \mathrm{Sect}(D_y \to G_L)$

$(L^{\times})^{\wedge}$-torsor $\rightsquigarrow$ canonical $L^{\times}$-torsor of non-zero elts $\in \omega_y$

the base of section assoc. to the tang. base pt θ

$\ddot{Y}_L \rightsquigarrow \ddot{y}_x$ $\quad O_L$-submodule $\hat{\omega}_y \subset \omega_y$ $\rightsquigarrow$ sub. gp $(L^{\times})^{\wedge}$

mod μ: $O_L^{\times}$ generator of $\hat{\omega}_y$ torsor of

(iii)]
4.1]

<u>Def 7.3</u> We call this canonical reduction of the $(L^\times)^\wedge$-torsor $\mathrm{Sect}(D_y \to G_C)$ to the can. $O_L^\times$-torsor <u>the canonical integral str. of D_y</u>

$s \in \mathrm{Sect}(D_y \to G_C)$ <u>compat w/ ↑</u>

$\Leftrightarrow$ s comes from a sect. of the can. $O_L^\times$-torsor

$L^\times$-torsor obtained by the push-out of the can. $O_L^\times$-torsor via $O_L^\times \to L^\times$ <u>the canonical discrete str. of D_y</u>

$\hat{\mathbb{Z}}' := \hat{\mathbb{Z}}^{(p')}$ $\quad (O_L^\times)' := \mathrm{Im}(O_L^\times \to (L^\times) \otimes \hat{\mathbb{Z}}')$

$(O_L^\times)'$-torsor : push-out : via $O_L^\times \to (O_L^\times)'$ <u>the canonical tame integral str. of D_y</u>

(cf. [AbsSect, Def 4.1 (ii), (iii)])

ω_y : cotang. of Y at y

We also get a reduction

$(L^\times)^n$-torsor $\mathrm{Sect}(D_3 \to G_L) \to$ $\{\pm 1\}$-torsor (resp. $\mu_{2\ell}$-torsor)

$$\frac{\{\pm 1\}\text{-str. of } D_3}{s \in \mathrm{Sect}(D_3 \to G_L) \text{ compat w/}} \quad \left(\text{resp. } \frac{\mu_{2\ell}\text{-str. of } D_3}{}\right)$$

if s comes from a section of the $\{\pm 1\}$-torsor (resp. $\mu_{2\ell}$-torsor)

$\ddot{Y} \supset \ddot{\mathcal{U}} \ (\cong \widehat{\mathbb{G}}_m)$

$\uparrow$ $\quad$ $\cup$ cond.

$\ddot{U}^2 = U$ $\quad$ irred. cmp. labelled $\mathbb{D} \in \mathbb{Z}$

(nodes)

$$\begin{array}{ccc} \ddot{\mathcal{U}} & \to & \mathcal{U} \\ \cap & & \cap \\ \ddot{Y} & \to & Y \end{array}$$

classical Theta

$\ddot{U} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}^{\times}_{\ddot{\mathcal{U}}})$

<u>Lem 7.4</u> ([EtTh, Prop 1.4])

$$\ddot{\Theta}(\ddot{U}) := q_X^{-\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n q_X^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{U}^{2n+1} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}_{\ddot{\mathcal{U}}})$$

extends uniquely to a mero. fct. on $\ddot{Y}$.

$$q_X^{-\frac{1}{8}} q_X^{\frac{1}{2}(n+\frac{1}{2})^2} = q_X^{\frac{n(n+1)}{2}}$$

cf. /$\mathbb{C}$ $\quad$ $q := e^{2\pi i \tau}$, $\ddot{U} = e^{\pi i z}$

$$\vartheta_{1,1}(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+\frac{1}{2})^2 + 2\pi i \tau (z+\frac{1}{2})(n+\frac{1}{2})} = \frac{1}{i} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{U}^{2n+1}$$

We also need a reduction

$(\mathbb{1}^{\times})^n$-torsor $\quad$ Sect$(\mathbb{D}_3 \to \mathbb{G}_c)$ $\quad$ to $\{\pm 1\}$-torsor (resp. $\mu_{2\ell}$-torsor)

(1). $\ddot{\Theta}(\ddot{U})$ has zeroes of order $=1$ at cusps of $\ddot{Y}$ & no other zeroes

poles of order $= j^2$ on the irred. comp. labelled j & no other poles

i.e. $\mathrm{div}(\ddot{\Theta}|_{\infty} = 0)$,

(2). $a \in \mathbb{Z}$ $\quad \ddot{\Theta}(\ddot{U}) = -\ddot{\Theta}(\ddot{U}^{-1})$, $\quad \ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$

$$\ddot{\Theta}(q_X^{\frac{a}{2}}\ddot{U}) = (-1)^a q_X^{-\frac{a^2}{2}}\ddot{\Theta}(\ddot{U})$$

(3). The classes $\mathcal{O}^\times_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}$ are precisely the Kummer classes assoc. to an $\mathcal{O}^\times_{\ddot{K}}$-mult. of the reg. fct $\ddot{\Theta}(\ddot{U})$ on $\ddot{Y}$

In particular, for a non-cuspidal pt $y \in \ddot{Y}(L)$ $\quad L/\ddot{K}$ fin,

$\mathcal{O}^\times_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}|_y \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$ lies in $L^\times \subset (L^\times)^\wedge$ & equal to $\mathcal{O}^\times_{\ddot{K}} \cdot \ddot{\Theta}(y)$

(3). The classes $\mathcal{O}^{\times}_{\ddot{K}}\ddot{\eta}^{\Theta}$ are precisely the Kummer classes assoc. to an $\mathcal{O}^{\times}_{\ddot{K}}$-mult. of the reg. fct $\hat{\Theta}(\ddot{U})$ on $\ddot{Y}$

In particular, for a non-cusp. pt $y \in \ddot{Y}(L)$ $L/\ddot{K}$ fin,

$$\mathcal{O}^{\times}_{\ddot{K}}\ddot{\eta}^{\Theta}\Big|_{y} \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^{\times})^{\wedge}$$

lies in $L^{\times} \subset (L^{\times})^{\wedge}$ & equal to $\mathcal{O}^{\times}_{\ddot{K}}\ddot{\Theta}(y)$

(4). For a cusp $y \in \ddot{Y}(L)$ w/ $L/\ddot{K}$ fin.

$D_y \subset \Pi_{\ddot{Y}}$ cusp. decp. at y

Take a sect. $s: G_L \hookrightarrow D_y$ compat. w/ the can. int. str. of D_y

$\rightsquigarrow \partial \in \hat{\omega}_y$ a generator

$$\mathcal{O}^{\times}_{\ddot{K}}\ddot{\eta}^{\Theta}\Big|_{s(G_L)} \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^{\times})^{\times}$$

lies in $L^{\times}$ $(\subset (L^{\times})^{\wedge})$

equal to $\mathcal{O}^{\times}_{\ddot{K}} \cdot \dfrac{d\ddot{\Theta}}{\hat{\Theta}}(y)$

the value at y of the first der. of $\ddot{\Theta}$ at y by $\hat{\Theta}$

(In part indep. of the choice of the generator $\partial \in \hat{\omega}_y$)

...nsor

...tion

...ur
...funsor

theta

$(-) \to (-)$ ell

$(\,)$

$q^{1/2l} =: \underline{q}$

$\Theta^{1/l} =: \underline{\Theta}$

(4). For a cusp $y \in \ddot{Y}(L)$ w/ $L/\ddot{K}$ fin.

$D_y \subset \Pi_{\ddot{Y}}$ cusp. decomp. at y

Take a sect. $s : G_L \hookrightarrow D_y$ cusp. at...

$\{\ \theta \in \hat{W}_y$ a generat...

$\mathcal{O}_{\ddot{K}}^{\times} \cdot \eta^{\Theta} \big|_{s(G_L)} \in H^1(G_L, \mathcal{O}$ ...

lies in $L^{\times}$ $(C(L^{\times})^{\wedge}$ ...

equal to $\mathcal{O}_{\ddot{K}}^{\times} \cdot$ ...

We call the classes in $\mathcal{O}_K^{\times} \cdot \eta^{\Theta}$

<u>étale theta function</u> in light of the above relationship of the values of theta fct & the restr. of these classes to G_L via pts

an autom ι of Π_Y^{tp}

lying over the action "-1" underlying ell. curve of X

fixes the irred. cmp. of Y labelled 0

<u>inversion autom.</u> of Π_Y^{tp}

Lem 7.5 ([EtTh, Prop 1.5])

$\Delta_\Theta \subset (\Delta^{t\text{-}p}_{\ddot{Y}})^\Theta \subset (\Pi^{t\text{-}p}_{\ddot{Y}})^\Theta$

(1). Leray

$$E_2^{a,b} = H^a\big((\Delta^{t\text{-}p}_{\ddot{Y}})^{\text{ell}}, H^b(\Delta_\Theta, \Delta_\Theta)\big) \Rightarrow H^{a+b}\big((\Delta^{t\text{-}p}_{\ddot{Y}})^\Theta, \Delta_\Theta\big)$$

$$E_2^{a,b} = H^a\big(G_{\ddot{K}}, H^b((\Delta^{t\text{-}p}_{\ddot{Y}})^\Theta, \Delta_\Theta)\big) \Rightarrow H^{a+b}\big((\Pi^{t\text{-}p}_{\ddot{Y}})^\Theta, \Delta_\Theta\big)$$

deg at E_2 $\rightsquigarrow$

$$H^1\big((\Pi^{t\text{-}p}_{\ddot{Y}})^\Theta, \Delta_\Theta\big) = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 \supset 0$$

$$\mathrm{Fil}^0/\mathrm{Fil}^1 \cong H_{\mathrm{an}}(\Delta_\Theta, \Delta_\Theta) = \hat{\mathbb{Z}}$$

$$\mathrm{Fil}^1/\mathrm{Fil}^2 = H_{\mathrm{an}}\big((\Delta^{t\text{-}p}_{\ddot{Y}})^\Theta/\Delta_\Theta, \Delta_\Theta\big) = \hat{\mathbb{Z}}\cdot \log(\ddot{U}) \cong \hat{\mathbb{Z}}(1)$$

$$\mathrm{Fil}^2 \cong H^1(G_{\ddot{K}}, \Delta_\Theta) \cong H^1(G_{\ddot{K}}, \hat{\mathbb{Z}}(1)) \overset{\text{id}}{\cong} (\ddot{K}^\times)^\wedge$$

We call the classes $\mathcal{O}^\times_{\ddot{K}}\cdot\eta^\Theta$

(2). $\forall\, \ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta) \leftarrow H^1((\Pi^{tp}_{\ddot{Y}})^{\Theta}, \Delta_\Theta)$

theta class

$$\ni\ \ddot{\eta}^{\Theta} \mapsto \mathrm{Fil}^0/\mathrm{Fil}^1 = \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) \ni \mathrm{id}$$

$$\mathcal{O}^{\vee}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \subset H^1((\Pi^{tp}_{\ddot{Y}})^{\Theta}, \Delta_\Theta)$$

consider additively $\ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})$

$$a \in \mathbb{Z} \cong \underline{\mathbb{Z}} = \Pi^{tp}_{X}/\Pi^{tp}_{\ddot{Y}} \curvearrowright \ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

$$\mapsto \ddot{\eta}^{\Theta} - 2a\log(\ddot{U}) - \frac{a^2}{2}\log(q_X) + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

▷ inversion automorphism of $\Pi_{\ddot Y}^{\pm\gamma}$

$$\iota(\ddot\eta^\Theta + \log(\mathcal{O}^\times_{\ddot F})) = \ddot\eta^\Theta + \log(\mathcal{O}^\times_{\ddot F})$$

$$\iota(\log(\ddot U) + \log(\mathcal{O}^\times_{\ddot F})) = -\log(\ddot U) + \log(\mathcal{O}^\times_{\ddot F})$$

$$H^1((\Pi^{\pm\gamma}_{\ddot Y})^\Theta, \Delta_\Theta) \to H^1((\Delta^{\pm\gamma}_{\ddot Y})^\Theta, \Delta_\Theta)^{G_{\ddot F}}$$

$$[\ddot\Theta(\ddot U)] \subset H^1((\Delta^{\pm\gamma}_{\ddot Y})^\Theta, \Delta_\Theta)$$

$$\to H^1(\Delta_\Theta, \Delta_\Theta) = \hat{\mathbb{Z}}$$

$$\downarrow$$

$$1$$

(2) $\Leftarrow$ fct eq.

Prop 7.6 (Anabelian Rigidity of the Étale Theta Function [EtTh, Th 1.6])

X (resp. ${}^{\dagger}X$) : smooth log-curve of type $(1,1)$ / $K \overset{fin}{\supset} \mathbb{Q}_p$ (resp. ${}^{\dagger}K \supset_{fin}$)

s.t. it has stable mod. over O_K (resp. $O_{{}^{\dagger}K}$)

& special fiber is sm. plan. geom. irred., node rational.

Similar obj's for ${}^{\dagger}X$, we use notation ${}^{\dagger}(\ )$

$\gamma : \Pi_X^{tp} \xrightarrow{\sim} \Pi_{{}^{\dagger}X}^{tp}$ isom of top. gps

(1), $\gamma(\Pi_{\ddot{Y}}^{tp}) = \Pi_{{}^{\dagger}\ddot{Y}}$

$|^{\Theta}, \Delta_G|^{G_K}$

$\Delta_\Theta|$

$\Delta_\Theta, \Delta_\Theta| = \hat{\mathbb{Z}}$

induces

$\Delta^{\dagger\mathrm{tm}}_{Y}|^{\Theta}, \Delta_\Theta|^{G_{\ddot K}}$

$|^{\Theta}, \Delta_\Theta|$

$H^1(\Delta_\Theta, \Delta_\Theta) = \hat{\mathbb{Z}}$

(2). $\gamma \overset{\text{induces}}{\rightsquigarrow} \Delta_\Theta \xrightarrow{\sim} {}^\dagger\Delta_\Theta$ compatible

$$H^1(G_{\ddot K}, \Delta_\Theta) \cong H^1(G_{\ddot K}, \hat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot K^\times)^\wedge \twoheadrightarrow \hat{\mathbb{Z}}$$

$$H^1(G_{{}^\dagger\ddot K}, {}^\dagger\Delta_\Theta) \cong H^1(G_{{}^\dagger\ddot K}, \hat{\mathbb{Z}}(1)) \cong ({}^\dagger\ddot K^\times)^\wedge \twoheadrightarrow \hat{\mathbb{Z}}$$

(3). $\gamma^* : H^1(\Pi^{\mathrm{tm}}_{\ddot Y}, \Delta_\Theta) \cong H^1(\Pi^{\mathrm{tm}}_{{}^\dagger\ddot Y}, {}^\dagger\Delta_\Theta)$

$$\mathcal{O}^{\times}_{\ddot K}\, \ddot\eta^{\Theta} \xrightarrow[\text{under}]{} \text{some } {}^\dagger\underline{\eta} \in \Pi^{\mathrm{tm}}_{{}^\dagger X} / \Pi^{\mathrm{tm}}_{{}^\dagger Y}\text{-conj. of } \mathcal{O}^{\times}_{{}^\dagger \ddot K}\, {}^\dagger\ddot\eta^{\Theta}$$

Prop 7.6 (Anabelian Rigidity of the Étale Theta Function [EtTh, Th 1.6])

X (resp. ${}^\dagger X$): smooth ... K^{fin} ...

$H^1(G_{k_i}, \Delta_\Theta) \cong H^1(G_{k_i}, \mathbb{Z}(1)) \to (k^\times)$

$\vartheta(t)$

$\shortparallel$

$t^{-\frac{1}{2}}\vartheta(\frac{1}{t})$

cf. Jacobi's identity

[IUTchI], upper half plane

[II HA end.] fnct. e.g.

[III log-shell] analy. cont.

\divideontimes

$\times \quad \div$

$\frac{l-1}{2} =: l^*$

$\frac{l\pm1}{2} =: l^\pm$

$\shortparallel$

$|\mathbb{F}_l|$

$\mathbb{F}_l^* := \mathbb{F}_l^\times/\{\pm1\}$, $\mathbb{F}_l^{\times\pm}$, $\mathbb{F}_l^{\circledast}$

(2), $\ddot{v}^\Theta$

theta

$\mathcal{O}^\times_{k_i}$

consider add

proof) $f(\Delta_X^{trop}) = \Delta_{+X}^{trop}$ by Lem 6.2

$f(\Delta_Y^{trop}) = \Delta_{+Y}^{trop}$ ← discreteness of $\frac{\mathbb{Z}}{+\mathbb{Z}}$

f: comp. decomp. → comp. decomp. Lem 6.1

$\ddot{Y} \to Y$ doubling

$$\begin{array}{ccc} \hat{C}_{sym} & \xrightarrow{(2)} & \hat{C}_{sym} \\ \downarrow & & \\ 1 \leftarrow \text{comp.} & & \end{array}$$

$f(\Pi_{\ddot{Y}}^{trop}) = \Pi_{+\ddot{Y}}^{trop}$

(2). $\gamma(\Delta_X^{\mathrm{tpd}}) = {}^{\dagger}\Delta_X^{\mathrm{tpd}}$

$\rightsquigarrow \gamma(\Delta_\Theta) = {}^{\dagger}\Delta_\Theta$ wrt of (2) ∈ Cor 6.7

up to int'l

Prop 2.1 (5), (6)

(3) After taking some $\Pi_X^{\mathrm{tpd}}/\Pi_Y^{\mathrm{tpd}} \cong \hat{\mathbb{Z}}$-conj., we may assume that
$\gamma: \Pi_Y^{\mathrm{tpd}} \xrightarrow{\sim} {}^{\dagger}\Pi_Y^{\mathrm{tpd}}$ is compat. w/ suitable inversion autom.

$\rightsquigarrow \gamma : \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) \longrightarrow \mathrm{Hom}({}^{\dagger}\Delta_\Theta, {}^{\dagger}\Delta_\Theta)$ (cf. [SemiAnbd, Th 6.8 (iii)], [AbsSect, Th 2.3]) (cf. (GC))

$\mathbb{F}_l^{\times}/\{\pm 1\}$ $\qquad$ $\mathbb{F}_l^{\times}/\{\pm 1\}$

$\mathrm{id} \longmapsto \mathrm{id}$

bet. Loc's

cal' again

fixed by γ (resp. ${}^{\dagger}\gamma$) up to $\mathcal{O}_{\bar{K}}^{\times}$-mult. (resp. $\mathcal{O}_{{}^{\dagger}\bar{K}}^{\times}$-mult.) by Cor 7.5 (2)

$\rightsquigarrow \ddot{\eta}^{\Theta}$ (resp. ${}^{\dagger}\ddot{\eta}^{\Theta}$) up to a $(\ddot{K}^{\times})^{\wedge}$-mult. (resp. $({}^{\dagger}\ddot{K}^{\times})^{\wedge}$-mult.)

Aetermin[illegible]

It suffices to reduce this $(\ddot{K}^{\times})^{\wedge}$-indet.

(resp. $({}^{\dagger}\ddot{K}^{\times})^{\wedge}$-indet.) to

$\ddot{O}^{\times}_{K}$-indet.

(resp. $\ddot{O}^{\times}_{{}^{\dagger}K}$-indet.)

From now on, we assume

(1). $\ddot{K} = K$,

(2). X w/ marked pt admits a K-core $X \twoheadrightarrow C := X /\!/ \{\pm 1\}$

quoti. in the sense of stacks

(3). $\sqrt{-1} \in K$

$\ddot{X} \twoheadrightarrow X$ Gal cov., deg $= 4$ dtd by mult.-by-2 $\left(\mathbb{G}_m^{\mathrm{rig}}/q_X^{\mathbb{Z}} \xrightarrow{2} \mathbb{G}_m^{\mathrm{rig}}/q_X^{\mathbb{Z}} \right)$

$\ddot{\mathfrak{X}} \to \mathfrak{X}$ model

$\ddot{X} \twoheadrightarrow C$ Galois w/

$\mathrm{Gal}(\ddot{X}/C) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$

... of
stacks

Choose $\sqrt{-1}, \sqrt{q}_X \in \overline{K}$

(i) $\tau \curvearrowright \sqrt{-1} \in K$ $\quad \ddot{\eta}^{\Theta}$

4-tors. μ^{*}

$\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2} \subset H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta)$

$\underline{\mathbb{Z}} \times \mu_2$ - orbit of $\ddot{\eta}^{\Theta}$

$\cong$

$\Pi^{tp}_X / \Pi^{tp}_{\ddot{Y}}$

Def (?) (cf. [EtTh, Def 1.9])

(ii) $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}\big|_{\tau}, \ \ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}\big|_{\tau^{-1}} \subset K^\times$

standard set of values of $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$

(2). there are two values in $K^\times$ of max. valuations of
zero and set of values of $\ddot{\eta}^{\Xi \times \mu_2}$

$$\left(\begin{array}{l} \text{Note } \ddot{\omega}\left(q_{b_x}^{\frac{a}{2}}\sqrt{-1}\right) = q_{b_x}^{-\frac{a^2}{2}}\,\ddot{\omega}\left(\sqrt{-1}\right) \quad \text{by Lem 7.4 (2)} \\ \ddot{\omega}\left(-q_{b_x}^{\frac{a}{2}}\sqrt{-1}\right) = -\ddot{\omega}\left(q_{b_x}^{\frac{a}{2}}\sqrt{-1}\right) \end{array} \right)$$

If they are equal to ± 1, $\Rightarrow$ $\ddot{\eta}^{\Theta, \Xi \times \mu_2}$: of standard type

Lem 7.8 ([EtTh, Prop 1.8])

$C = X /\!/ \pm 1$ $({}^{\dagger}C = {}^{\dagger}X /\!/ \pm 1)$ over K (resp. ${}^{\dagger}K$) $/^{\text{fin}} \mathbb{Q}_p$

resp. smooth hyp-cur. $\sqrt{-1} \in K$ (resp. ${}^{\dagger}K$)

$${}^{\dagger}(-)$$

$$\gamma : \Pi_C^{tp} \xrightarrow{\sim} \Pi_{{}^{\dagger}C}^{tp} \Rightarrow \Pi_X^{tp} \xrightarrow{\sim} \Pi_{{}^{\dagger}X}^{tp}, \quad \Pi_{\ddot{X}}^{tp} \xrightarrow{\sim} \Pi_{{}^{\dagger}\ddot{X}}^{tp}$$

induces

$$\Pi_{\ddot{Y}}^{tp} \xrightarrow{\sim} \Pi_{{}^{\dagger}\ddot{Y}}^{tp}$$

Prop 7.9 (Constant Multiple Rigidity of the Étale Theta Function, [EtTh, Thm 1.10])

Let $C = X /\!/ \{\pm 1\}$ (resp. ${}^{\dagger}C = {}^{\dagger}X /\!/ \{\pm 1\}$)

$K = \mathrm{cne}$ smooth log-orbicurve $/ K$ (resp. ${}^{\dagger}K$) $\supset \mathbb{Q}$ fin.

$K \ni \sqrt{-1}$, ${}^{\dagger}K \ni \sqrt{-1}$

${}^{\dagger}C \rightsquigarrow \Theta\,;\ {}^{\dagger}(-)$

$\gamma : \Pi_C^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}C}^{\mathrm{tp}}$ isom. of top. gps

(by Lem 7.8 $\rightsquigarrow \gamma : \Pi_X^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{{}^{\dagger}X}^{\mathrm{tp}}$)

Assume γ maps the subset $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{tp}}, \Delta_\Theta)$

to ${}^{\dagger}\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2} \subset H^1(\Pi_{{}^{\dagger}\ddot{Y}}^{\mathrm{tp}}, {}^{\dagger}\Delta_\Theta)$

(1). γ preserves the property that $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ is of std type.

$\hookrightarrow$ uniquely det's this collection of classes

(2). $\gamma \rightsquigarrow K^\times \xrightarrow{\sim} {}^\dagger K^\times \subset ({}^\dagger K^\times)^\wedge \cong H^1(G_{{}^\dagger K}, {}^\dagger\Delta_\Theta) \subset H^1(\Pi^{tp}_{{}^\dagger C}, {}^\dagger\Delta_\Theta)$

$\cap$

$(K^\times)^\wedge \cong H^1(G_K, \Delta_\Theta) \subset H^1(\Pi^{tp}_{C}, \Delta_\Theta)$

γ maps the std rat'l functions of $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ to $\vdots$ of ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$

(3). $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ (hence ${}^\dagger\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ as well by (1)) is of std type. K (hence ${}^\dagger K$) res. char >2

$\Rightarrow$ $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}\times\mu_2}$ (resp. ${}^\dagger(-)$) determines a $\{\pm 1\}$-str. on $(K^\times)^\wedge$-torsor (resp. ${}^\dagger(-)$) at the unique cusp of C (resp. ${}^\dagger C$) which is compat w/ the ram. int. str., and it is preserved by γ.

... (resp. $\dagger(-)$) determines a $\{\pm 1\}$-str. on $(K^\times)$-torsor (resp. $\dagger(-)$)
at the unique cusp of C (resp. $\dagger C$) which is compat. w/
the cam. int. str. and it is preserved by γ.

☺: (1), (3) $\Leftarrow$ (2)

(2): γ induces an isom. of dual graph of $\ddot{Y}$ $\to$: : $\dagger\ddot{Y}$ (by Prop 6.5)

éll. cusp'ization (Th 3.7) $\exists$ γ: decomp. gp of pts of $\ddot{Y}$ lying over τ

$\mapsto$: : lying over $\tau^{\pm 1}$

$\rightsquigarrow$ (2) //

§7.3 l-th Root of Étale Theta Function

X : smooth log-curve of type $(1,1)$ / a field K of char $=0$

$\supset F_l$

Assume X admits K-core

$$1 \to \Delta_X \to \Pi_X \to G_K \to 1$$

$$\Delta_X^{ell} := \Delta_X^{ab}, \quad \Delta_X^{\Theta} := \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]]$$

$$\Delta_\Theta := \mathrm{Im}(\Lambda^2 \Delta_X^{ell} \to \Delta_X^{\Theta}) \qquad 1 \to \Delta_\Theta \to \Delta_X^{\Theta} \to \Delta_X^{ell} \to 1$$

$$\Pi_X^{\Theta} := \Pi_X / \ker(\Delta_X \to \Delta_X^{\Theta})$$

...ches

$t_\eta^{\Theta, \mathbb{Z}\times\mu}$

...g. t_η)

...etermined by Y,

$l > 2$ prime

$\overline{\Delta}_X := \Delta_X^\Theta /$ the subgp gen. by l-th powers of elts of Δ_X^Θ

($l \neq 2$ normal)

$\overline{\Delta}_\Theta := \mathrm{Im}(\Delta_\Theta \to \overline{\Delta}_X)$ $(\cong \mathbb{Z}/l(1))$

$\overline{\Delta}_X^{ell} := \overline{\Delta}_X / \overline{\Delta}_\Theta$, $\overline{\Pi}_X := \Pi_X / \mathrm{Ker}(\Delta_X \to \overline{\Delta}_X)$

$\overline{\Pi}^{ell} := \overline{\Pi}_X / \overline{\Delta}_\Theta$

free $\mathbb{Z}/l$-module of rank $= 2$

ι: unique cusp of X, $I_\iota \subset D_\iota$

§7.3 l-th Root of Étale Theta Function

$$
\begin{array}{ccc}
D_X & \hookrightarrow & \Pi^{\Theta}_X \\
\vee & & \cup \\
I_X & \xrightarrow{\sim} & \Delta_\Theta
\end{array}
$$

$$\overline{D}_X := \mathrm{Im}(D_X \to \overline{\Pi}_X)$$

$$1 \to \underset{\substack{\wr\!\wr \\ \mathbb{Z}/\ell(1)}}{\overline{\Delta}_\Theta} \to \overline{D}_X \to G_k \to 1$$

<u>Assumption (1)</u> Choose a quot. $\overline{\Pi}^{\mathrm{ell}}_X \twoheadrightarrow Q$ ← free $\mathbb{Z}/\ell$-mod of rk $= 1$

s.t.

$$
\begin{array}{ccc}
\overline{\Pi}^{\mathrm{ell}}_X & \twoheadrightarrow & Q \\
\cup & \nearrow\!\!\!\!\!\rightarrow & \\
\overline{\Delta}^{\mathrm{ell}}_X & & \uparrow \\
\cup & \nearrow_{\text{triv.}} & \\
D_X & &
\end{array}
$$

$$\begin{array}{ccc} D_x & \hookrightarrow & \Pi^{\Theta}_X \\ \vee & & \cup \\ I_x & \xrightarrow{\sim} & \Delta_\Theta \end{array}$$

$$\overline{D}_x := \mathrm{Im}(D_x \to \overline{\Pi}_X)$$

$$1 \to \overline{\Delta}_\Theta \to \overline{D}_x \to G_{k_1} \to 1$$

$$\left(\begin{array}{c} \overline{\Delta}_\Theta \\ \cong \\ \mathbb{Z}_\ell(1) \end{array} \right)$$

<u>Assumption(1)</u> Choose a quot. $\overline{\Pi}^{ell}_X \twoheadrightarrow Q$, free $\mathbb{Z}_\ell$-mod of $\mathrm{rk}=1$

s.t.

$$\begin{array}{ccc} \overline{\Pi}^{ell}_X & \twoheadrightarrow & Q \\ \cup & \nearrow & \\ \Delta^{ell}_X & & \\ \cup & & \\ D_x & \xrightarrow{\text{triv}} & Q \end{array}$$

$\overline{D}_x \simeq \overline{\Pi}_{\underline{X}} / \mathrm{Im}(s_?)$ $/ G_k$

acts by

$\underline{?} \curvearrowright Q \quad (-1)$

$(l \neq 2)$

$\forall$ gen. $\mathrm{Gal}(\underline{X}/X)$ does not descend to an autom. of $\underline{C}$ over C

$\rightsquigarrow \underline{C} \to C$ not Galois, cf. [EtTh, Rem 2.1.1]

$\overline{\Pi}_C := \Pi_C / \ker(\Pi_X \twoheadrightarrow \overline{\Pi}_X)$ $\quad \overline{\Delta}_C$

$\overline{\Pi}_{\underline{C}} := \Pi_{\underline{C}} / \ker(\Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}})$ $\quad \overline{\Delta}_{\underline{C}}$ $\quad , \overline{\Delta}_C^{\,ell} := \overline{\Delta}_C / \ker(\overline{\Delta}_X \twoheadrightarrow \overline{\Delta}_X^{\,ell})$

Assumption (2) Choose $\varepsilon_{\underline{C}} \in \overline{\Delta}_{\underline{C}}$ which lifts the $(1\neq)$ elt $\in \mathrm{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$

$\rightsquigarrow \varepsilon_{\underline{C}} \curvearrowright$ cj. $\overline{\Delta}_{\underline{X}}$ $\quad$ free of rk $= 2$ / $\mathbb{Z}_l$

eigenvalue -1 (resp. $+1$)

$\rightsquigarrow \overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$ (resp. $\overline{\Delta}_\Theta$)

$\rightsquigarrow \overline{\Delta}_{\underline{X}} \cong \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \times \overline{\Delta}_\Theta$ (cf. [EtTh, Prop 2.2 (i)])

compat w/ conj. action of $\overline{\Pi}_{\underline{X}}$ ($\leftarrow$ $\iota_{\underline{L}}$ (centers) $\overline{\Pi}_{\underline{X}}$-conj.)

$s_l : \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \hookrightarrow \overline{\Delta}_{\underline{X}}$ the splitting of $\overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$

$\overline{D}_x \xrightarrow{\sim} \overline{\Pi}_{\underline{X}} / \mathrm{Im}(s_l)$ $/ G_K$

acts by

Assumption(3) Choose any elt $s^{\mathrm{Assump}(3)} \in \mathrm{Sect}(\overline{D}_X \twoheadrightarrow G_k)$ $\quad H^1(G_k, \overline{\Delta}_\Theta) \cong k^\times/(k^\times)^\ell$ -torsor

$$\rightsquigarrow \Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}} / I_n(\varsigma_2) \xrightarrow{\sim} \overline{D}_X \twoheadrightarrow \overline{D}_X / s^{\mathrm{Assump}(3)}(G_k) \cong \overline{\Delta}_\Theta$$

$$\rightsquigarrow \underline{\underline{X}} \to \underline{X}$$

$$\mathrm{Gal} = \overline{\Delta}_\Theta \ (\cong \mathbb{Z}/\ell\mathbb{Z})$$

$\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_{\underline{X}}$, $\overline{\Pi}_{\underline{\underline{X}}} \subset \overline{\Pi}_{\underline{X}}$

$\overline{\Delta}_{\underline{\underline{X}}} = I_n(\varsigma_2)$

$\overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\underline{\underline{X}}} \cdot \overline{\Delta}_\Theta$

$\ell > 2$ pr

in Anab. $\Pi_C^{tp} \xrightarrow{\sim} \Pi_{+C}^{tp}$, $\Pi_{\underline{C}}^{tp} \xrightarrow{\sim} \Pi_{+\underline{C}}^{tp}$,

$\Pi_X^{tp} \xrightarrow{\sim} \Pi_{+X}^{tp}$, $\Pi_{\underline{X}}^{tp} \xrightarrow{\sim} \Pi_{+\underline{X}}^{tp}$, $\Pi_{\ddot{Y}}^{tp} \xrightarrow{\sim} \Pi_{+\ddot{Y}}^{tp}$

$I_x \cong \Delta_\Theta \twoheadrightarrow \overline{\Delta}_\Theta \quad \sim \underline{\underline{X}} \to \underline{X}$

tot. ram. at the cusps

image of $\epsilon_{\underline{C}}$ in $\overline{\Delta}_X / \overline{\Delta}_{\underline{X}}$ char'zed as the unique coset of $\overline{\Delta}_C / \overline{\Delta}_{\underline{X}}$

which lifts the $(1 \neq)$ elt $\in \overline{\Delta}_C / \overline{\Delta}_X$

normalizes the subgp $\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_C$

(cf. [EtTh, Prop 2.2 (ii)]) $\left(\Delta_{\underline{X}} / \overline{\Delta}_{\underline{\underline{X}}} \cong \overline{\Delta}_\Theta \right.$

eigenvalue $= 1$ $\left. \curvearrowright \overline{\Delta}_\Theta \right)$

Lem 7.10 ([EtTh, Prop 2.4])

$\underline{\underline{X}}$ (resp. ${}^{\dagger}\underline{\underline{X}}$) smooth log curve / K (resp. ${}^{\dagger}K$) $\supseteq_{\text{fin}} \mathbb{Q}_p$

constructed as above.

${}^{\dagger}\underline{\underline{X}} \rightsquigarrow {}^{\dagger}(-)$ X (resp. ${}^{\dagger}X$) has stable red over O_K (resp. $O_{{}^{\dagger}K}$)

sp. fiber singular, geom. irred.

w/ nat. node

$\gamma : \Pi^{tp}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}\underline{\underline{X}}}$ isom of tp. gps

$\xrightarrow[\text{in Anab}]{} \Pi^{tp}_{C} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}C}$, $\Pi^{tp}_{\underline{C}} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}\underline{C}}$,

$\Pi^{tp}_{X} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}X}$, $\Pi^{tp}_{\underline{X}} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}\underline{X}}$, $\Pi^{tp}_{\ddot{Y}} \xrightarrow{\sim} \Pi^{tp}_{{}^{\dagger}\ddot{Y}}$

local

$X \quad X$

bad $=$ good

global

$\underline{X}_K, \quad \underline{C}_K$

$\uparrow \quad \uparrow$

$\boxplus \quad \boxtimes$

germ. arith

$\mathbb{F}_\ell^{\rtimes\pm}, \quad \mathbb{F}_\ell^{*}$

HA -eval. NF Kum

(cyc-synchro

⊚: Kum

cyclotomic log-link

gen.

Π

diag

hori core

$\mathbb{F}_\ell$ — $\mathbb{F}_\ell$

$\mathbb{F}_\ell^{*}$ — $\mathbb{F}_\ell^{*}$

$\mathbb{F}_\ell^{\rtimes\pm} = \mathbb{F}_\ell \rtimes \{\pm 1\}$

$\mathbb{F}_\ell^{\times}$

vert. core

cusp $\boxplus$

s.s. pts $\boxtimes$

cusps

Rem 7.10.1 ([EtTh, Rem 2.6.1])

$\mu_\ell \subset K$

Lem 7.10

$\mathrm{Aut}_K(\underline{X}) = \mu_\ell \times \{\pm 1\}$

$\mathrm{Aut}_K(\underline{\underline{X}}) = \mathbb{Z}/\ell\mathbb{Z} \rtimes \{\pm 1\}$

$\mathrm{Aut}_K(\underline{C}) = \{1\}$

$\underline{X} \to X$

$1, 2, \dots, \ell^* = \frac{\ell - 1}{2}$

$(\ell^*)!\ell^*$

$\{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \dots \subset \{1,\dots,\ell^*\}$

$(\ell^*)!$

for $\le$

Def 7.11 ([E+Th, Def 2.5])

Assume K : res. char > 2, $\ddot{K}_1 = K$

Make two assumptions

Assumption (4) $\quad \Pi_X^{ell} \twoheadrightarrow Q$ factors through the nat. quot. $\Pi_X \twoheadrightarrow \hat{\mathbb{Z}}$ det'd by the quot $\Pi_X^{top} \twoheadrightarrow \mathbb{Z}$, Y

Assumption (5) $\quad s^{Aug(3)} \in \mathrm{Sect}(\Pi_X \twoheadrightarrow G_K)$ is compat w/ the $\{\pm 1\}$-str. of Prop 7.9(3)

$\Theta, \ddot{Y}$

A smooth $\mathcal{O}_J$-scheme $\mathbb{K}$ is called

$\underline{\text{of type } (1, \mathbb{Z}/\ell\mathbb{Z})}$ (resp. $\underline{\text{of type } (1, (\mathbb{Z}/\ell)^{\oplus})}$)

if it is isom. to $\underline{X}$ (resp. $\underline{\underline{X}}$)

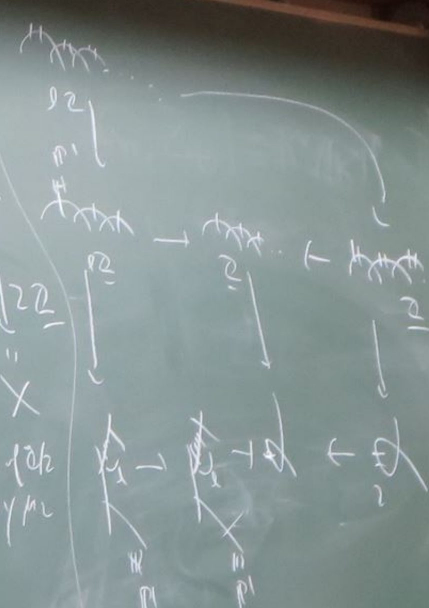
$\underline{\underline{\dot{Y}}} \to X$ (resp. $\underline{\underline{\ddot{Y}}} \to X$) the composite of the covering $\dot{Y} \to X$

w/ $\underline{\underline{X}} \to X$ (resp. $\ddot{Y} \to X$)

$$\begin{tikzcd}
\underline{\underline{Y}} \arrow[d, "\mu_2"] \arrow[rrr, bend left, "\mathbb{Z}_\theta (\cong \mathbb{Z}/\ell(1))"] & & & \\
\underline{\underline{Y}} \arrow[d, "\ell\mathbb{Z}"] \arrow[rr] & & Y \arrow[d, "\mathbb{Z}"] & Y \arrow[l, "\mu_2"'] \arrow[d, "2\mathbb{Z}"] \\
\underline{\underline{X}} \arrow[r, "\mathbb{Z}_\theta"'] & \underline{X} \arrow[r, "\mathbb{Q}"'] & X & X \arrow[l]
\end{tikzcd}$$

$\cong \mathbb{Z}/\ell(1)$ $\{\pm 1\}$ $\cong (\mathbb{Z}_\ell)$ $\{\pm 1\}$ ext'n of $\mathbb{Z}/2$ by μ_2

$$\mathbb{C} \xrightarrow[\text{non-cont}]{} \mathbb{C}$$



A smooth $\mathbb{G}_a$-scheme $\mathbb{K}$ is called special fibres

• X	: 1 irred. comp	$(\cong \mathbb{P}^1)$	1 cusp on it
• $\ddot{X}$	: 2 irred. "	$(\cong \mathbb{P}^1)$	2 cusps on each
• $\underline{X}$	: ℓ " "	$(\cong \mathbb{P}^1)$	1 " on each
• $\underline{\underline{X}}$	: ℓ " "	$(\ncong \mathbb{P}^1)$	1 " on each
• Y	: irred. comp $(\cong \mathbb{P}^1)$ parametrised by $\underline{Z}$		1 cusp on each
• $\ddot{Y}$	: " " $(\cong \mathbb{P}^1)$	by $\underline{Z}$	2 " "
• $\underline{Y}$	: " " $(\ncong \mathbb{P}^1)$	by $\ell\underline{Z}$	1 " "
• $\underline{\underline{\ddot{Y}}}$	: " " $(\ncong \mathbb{P}^1)$	by $\ell\underline{Z}$	2 " "

By Assump (4) $\rightsquigarrow$

$$\begin{tikzcd} \ddot{Y} \arrow[r] \arrow[dr, dashed] & X \\ & \underline{X} \arrow[u] \end{tikzcd}$$

$\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta)$

well-def'd up to a $\mathbb{Q}_l^{\times}$-mult.

$\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta \otimes \mathbb{Q}/\mathbb{Z})$ $\quad \Pi^{tp}_{\underline{X}} / \Pi^{tp}_{\ddot{Y}} \cong \underline{\mathbb{Z}} \times \mu_2$-what

(by Lem 7.5 (2)) $\quad$ can be regarded as alg's assoc. to $\Pi^{tp}_{\underline{X}}$

$\ddot{\eta}^{\Theta} \in H^1((\Pi^{tp}_{\ddot{Y}})^{\Theta}, \Delta_\Theta \otimes \mathbb{Q}/\mathbb{Z})$ $\quad (\Pi^{tp}_{\ddot{Y}})^{\Theta} \twoheadleftarrow \Pi^{tp}_{\ddot{Y}} \leftarrow D_x$

$\downarrow$

$H^1(D_x, \Delta_\Theta \otimes \mathbb{Q}/\mathbb{Z})$

$\circledast \in H^1(\bar{D}_x, \Delta_\Theta \otimes \mathbb{Q}/\mathbb{Z})$

$\in \ddot{\eta}^{\Theta} \in H^1(\bar{D}_x,$

(!)

special fiber

X ... ized curve $(\cong \mathbb{P}^1)$ 1 cusp on it

$$0 \to H^1(\bar{D}_x, \Delta_\theta \otimes \mathbb{Z}/\ell) \to H^1(D_x, \Delta_\theta \otimes \mathbb{Z}/\ell) \to H^1(\ell\Delta_\theta, \Delta_\theta \otimes \mathbb{Z}/\ell\mathbb{Z})$$

$$= \mathrm{Hom}(\ell\Delta_\theta, \Delta_\theta \otimes \mathbb{Z}/\ell)$$

$$a \longmapsto \eta^\theta \longmapsto 0$$

$s \in \mathrm{Sect}(\bar{D}_x \to G_K)$,

$$\bar{D}_x \ni \bar{g} \longmapsto g(s(\bar{g}))^{-1} \quad \text{1-cocycle} \rightsquigarrow a \in H^1(\bar{D}_x, \Delta_\theta \otimes \mathbb{Z}/\ell\mathbb{Z})$$

$$G \ni g \mapsto \bar{g}$$

$$\mathrm{Sect}(\bar{D}_x \to G_K) \xrightarrow[\text{map}]{} H^1(\bar{D}_x, \Delta_\theta \otimes \mathbb{Z}/\ell\mathbb{Z})$$

$$0 \to H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/\ell) \to H^1(D_x, \Delta_\Theta \otimes \mathbb{Z}/\ell) \to \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$$

$$H^1(G_K, \Delta_\Theta \otimes \mathbb{Z}/\ell) \leftarrow 0$$

$$K^\times/(K^\times)^\ell$$

$\mathrm{Sect}(\overline{D}_x \to G_K) \ni s$

$\mathrm{In}(s) \ni \eta^\Theta$

$H^1(\Delta_\Theta \otimes \mathbb{Z}/\ell, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

$= \mathrm{Hom}$

$\ni \mathrm{id}$

$H^1((\pi_1^{tp})^\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

by Lem 7, $s(2)$

$$g \mapsto g(s(\bar g)) = g(s(1))^{-1} = g \cdot 1^{-1} = g$$

$\Delta_\Theta \otimes \mathbb{Z}/\ell$

$\rightsquigarrow$ In, $\ni$ $\ddot\eta^\Theta$ differs from $\ddot\eta^\Theta \in H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/\ell)$ by an

$H^1(G_K, \Delta_\Theta \otimes \mathbb{Z}/\ell) \cong K^\times/(K^\times)^\ell$ -multiple

$s^{\text{Assump}(3)} \in \mathrm{Sect}(\overline{D}_x \to G_K)$

$$0 \to H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}/\ell) \to H^1(D_x, \Delta_\Theta \otimes \mathbb{Z}/\ell) \to H^1(\ldots$$

$\mathrm{Im}(\varsigma^{\mathrm{Assmp}(3)}) \in H^1(\overline{D}_\gamma, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

We can modify $\eta^\Theta \in H^1(\overline{D}_\gamma, \Delta_\Theta \otimes \mathbb{Z}/\ell)$ by a $K^\times$-multiple

well-def up to $(K^\times)^\ell$-multiple,

coincide w/ $\mathrm{Im}(\varsigma^{\mathrm{Assmp}(3)}) \in H^1(\overline{D}_\gamma, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

stronger claim also holds, i.e.

we can modify η^Θ by an $\mathcal{O}_K^\times$-multiple

well-def up to $(\mathcal{O}_K^\times)^\ell$-multiple to make it coincide w/ $\mathrm{Im}(\varsigma^{\mathrm{Assmp}(3)})$

$\varsigma^{\mathrm{Assmp}(3)}$ is compat w/ the $\{\pm 1\}$-str in the case

$\mathrm{Im}(\mathfrak{s}^{\mathrm{Assump}(3)}) \in H^1(\overline{\Pi}_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

We can modify $\ddot{\eta}^\Theta \in H^1(\overline{\Pi}_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$ by a $K^\times$-multiple
well-def up to $(K^\times)^\ell$-multiple,
coincide w/ $\mathrm{Im}(\mathfrak{s}^{\mathrm{Assump}(3)}) \in H^1(\overline{\Pi}_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$
stronger claim also holds, i.e.,
we can modify $\ddot{\eta}^\Theta$ by an $\mathcal{O}_K^\times$-mult
well-def up to $(\mathcal{O}_K^\times)^\ell$-multiple to make it coincide w/ $\mathrm{Im}(\mathfrak{s}^{\mathrm{Assump}(3)})$,
since $\mathfrak{s}^{\mathrm{Assump}(3)}$ is compat w/ the can. int. str. of Π_X by Assump(5).

As a cor., $\rightsquigarrow$ by modify $\ddot{\eta}^\Theta \in H^1((\Pi_{\ddot{Y}})^\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$
by a $\mathcal{O}_K^\times$-multiple
which is well-def up to $(\mathcal{O}_K^\times)^\ell$-multiple. (← not $\mathcal{O}_K^\times$-mult.)
we can and shall assume that
$\ddot{\eta}^\Theta = \mathrm{Im}(\mathfrak{s}^{\mathrm{Assump}(3)}) \in H^1(\overline{\Pi}_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$
(i.e. by the char. of $\underline{\underline{X}}$ $\quad s_\ell, \tau_\ell$
$\underline{\underline{X}} \twoheadrightarrow \underline{X}$ "the covering of" "taking ℓ-th root of Θ")

$0 \to H^1(\overline{D}_x, \Delta_\Theta \otimes \mathbb{Z}\ell) \to H^1(D_x, \Delta_\Theta \otimes \mathbb{Z}\ell$

η^Θ

$s \in \mathrm{Sect}(\overline{D}_x \to G_K)$,

$\overline{D}_x \ni g \mapsto g(s(\overline{g}))^{-1}$ ill-coagle

$\mathrm{Sect}(\overline{D}_x$

mono-theta env

cycl. rig.

const. multi rig.

disc. rig.

μ^Θ

θ

$\overline{\theta}$

Θ/μ

splitting

θ^{+}

$\mathcal{O}^\times$

Π_Y

μ^Θ